Student Workbook

Solving Systems by Row Reduction

Name: ______________________ Date: ______________ Class: ________

ESSENTIAL QUESTION

How does transforming an augmented matrix to row echelon form through systematic row operations give you the solution to a linear system, and what does it mean when you reach a row of all zeros?

Lesson Overview

The augmented matrix [A|b] is formed by appending the constant vector b to the coefficient matrix A. Applying elementary row operations transforms it toward row echelon form (REF) or reduced row echelon form (RREF), from which the system's solution can be read off or back-substituted.

Row Operations → REF / RREF → Solution

Aug. Matrix [A|b]: the coefficient matrix A with the constant vector b appended after a vertical bar.

Row Ops Swap two rows ($R_i \leftrightarrow R_j$); multiply a row by a nonzero k ($kR_i \to R_i$); add a multiple of one row to another ($R_i + kR_j \to R_i$).

REF / RREF REF: zero rows at the bottom, each pivot right of the one above, zeros below every pivot. RREF: also every pivot=1 and zeros above each pivot.

Algorithms Gaussian elimination reduces to REF then back-substitutes. Gauss-Jordan reduces all the way to RREF so the solution reads off directly.

Results Unique solution: every variable has a pivot. No solution: a row [0 0 0|k], k≠0. Infinitely many: a row of all zeros (free variable).

Worked Examples

EXAMPLE 1

Solve using Gaussian elimination: 2x + y = 5, x + 2y = 4.

- Augmented: [[2,1|5],[1,2|4]]. Swap $R_1 \leftrightarrow R_2$: [[1,2|4],[2,1|5]]
- $R_2 \to R_2 - 2R_1$: [[1,2|4],[0,−3|−3]]. $R_2 \to R_2 \div (-3)$: [[1,2|4],[0,1|1]]
- Back-substitute: y=1; x+2(1)=4 → x=2

Answer: (2, 1)

EXAMPLE 2

Solve: x + y + z = 6, 2x − y + z = 3, x + 2y − z = 2.

- $R_2 \to R_2 - 2R_1$: [0,−3,−1|−9]. $R_3 \to R_3 - R_1$: [0,1,−2|−4]. Swap $R_2 \leftrightarrow R_3$
- $R_3 \to R_3 + 3R_2$: [0,0,−7|−21]. $R_3 \to R_3 \div (-7)$: z=3
- Back-substitute: y−2(3)=−4 → y=2; x+2+3=6 → x=1

Answer: (1, 2, 3)

EXAMPLE 3

Identify the solution type: x + 2y = 3, 2x + 4y = 6.

- Augmented: [[1,2|3],[2,4|6]]. $R_2 \to R_2 - 2R_1$: [[1,2|3],[0,0|0]]
- Row of all zeros → dependent system (free variable)

Answer: Infinitely many solutions; y is free, x = 3 − 2y

EXAMPLE 4

Identify the solution type: x + y = 3, x + y = 5.

- Augmented: [[1,1|3],[1,1|5]]. $R_2 \to R_2 - R_1$: [[1,1|3],[0,0|2]]
- [0 0 | 2] means 0 = 2 → contradiction

Answer: No solution (inconsistent system)

EXAMPLE 5

Solve using Gauss-Jordan (RREF): x + 2y = 5, 3x + 5y = 13.

- Augmented: [[1,2|5],[3,5|13]]. $R_2 \to R_2 - 3R_1$: [[1,2|5],[0,−1|−2]]
- $R_2 \to -R_2$: [[1,2|5],[0,1|2]]. $R_1 \to R_1 - 2R_2$: [[1,0|1],[0,1|2]]
- RREF reached — read solution directly

Answer: x = 1, y = 2

Guided Practice

Work through each problem below. Use the hint if you get stuck — write your final answer on the last line.

1 Write the augmented matrix for: 3x − y = 7, 2x + 4y = 10. Then reduce to REF.

Hint: Swap rows if needed to get a leading 1 in position (1,1).

2 Solve using Gaussian elimination: x + 2y + z = 8, 2x + y − z = 3, x − y + 2z = 5.

Hint: Use $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$ to eliminate x from rows 2 and 3.

3 Solve using Gauss-Jordan (RREF): $2x + y = 7$, $x - 3y = -4$.

Hint: Reduce all the way to RREF so the solution can be read directly.

4 Determine the solution type: $2x - 4y + 6z = 4$, $x - 2y + 3z = 2$, $3x - 6y + 9z = 6$.

Hint: What happens when you eliminate x from rows 2 and 3?

5 Solve: $x + y - z = 2$, $2x + 3y + z = 11$, $x + 2y + 2z = 9$.

Hint: Systematically eliminate x, then y.

Key Vocabulary

Augmented Matrix	The coefficient matrix with the constant column appended, written [A|b].
Row Operations	Swap rows, multiply a row by a nonzero constant, or add a multiple of one row to another. All three preserve the solution set.
REF	Each pivot is to the right of the one above it; all entries below each pivot are 0.
RREF	REF with each pivot equal to 1 and all other entries in pivot columns equal to 0.
Pivot	The leading nonzero entry in a row; used to eliminate entries below (and above in RREF).
Free Variable	A variable without a pivot column; its presence indicates infinitely many solutions.

Check Your Understanding

Circle the letter of the correct answer for each question.

1 **The augmented matrix for $2x - y = 3$, $x + 4y = 7$ is:** Multiple Choice

A [[2,−1,3],[1,4,7]]

B [[2,−1|3],[1,4|7]]

C [[2,1],[−1,4],[3,7]]

D [[3,7],[2,−1]]

2 **A row of [0 0 0 | 5] in a reduced matrix means:** Multiple Choice

- **A** x=0, y=0, z=5
- **B** The system has infinitely many solutions
- **C** The system has no solution
- **D** z = 5

3 **Which is NOT a valid row operation?** Multiple Choice

- **A** Swap two rows
- **B** Multiply a row by 0
- **C** Add a multiple of one row to another
- **D** Multiply a row by −3

4 **A row of [0 0 0 | 0] in a reduced matrix means:** Multiple Choice

- **A** No solution
- **B** Unique solution
- **C** Infinitely many solutions (free variable)
- **D** The matrix is in RREF

5 **Gauss-Jordan elimination produces:** Multiple Choice

- **A** Row Echelon Form
- **B** Reduced Row Echelon Form
- **C** The inverse matrix
- **D** The determinant

Common Mistakes to Avoid

✗ Applying a row operation to only part of the row.

✓ A row operation applies to the ENTIRE row, including the augmented column. Never forget to update the constant column.

✗ Multiplying a row by 0 to create zeros.

✓ Multiplying by 0 destroys the row. Use $R_i + kR_j \to R_i$ to create zeros by adding a multiple of another row.

✗ Stopping at REF and forgetting to back-substitute.

✓ After reaching REF, use back-substitution from the bottom row up to find all variable values.

✗ Confusing a row of zeros [0 0 0 | 0] with [0 0 0 | k] ($k \neq 0$).

✓ [0 0 0 | 0] means a free variable (infinitely many solutions). [0 0 0 | k] means no solution.

Math Tips

★ Always aim to get a leading 1 in each pivot position. Swap rows or divide a row to create a 1 before eliminating below.

★ Work column by column, left to right. Eliminate all entries below the current pivot before moving to the next column.

★ Label each row operation clearly (e.g., $R_2 \to R_2 - 2R_1$). This makes it easy to check your work and find errors.

★ For RREF, after getting REF, work back up: use each pivot to eliminate entries ABOVE it as well.

★ The number of pivots = the number of independent equations = the number of variables with unique values.